

$$a^{p-1} b_1 p^2 \equiv 0 \pmod{p^3},$$

thus,  $b_1 \equiv 0 \pmod{p},$  (13)

$$b_1 = b_2 p.$$

We make a substitution of (13) into (12) to get

$$\sum_{k=0}^p a^k j^{p-k} b_2^{p-k} (p^2)^{p-k} \binom{p}{k} = c^p \quad (14)$$

and take the imaginary part modulo  $p^4$ . This gives

$$a^{p-1} b_2 p^3 \equiv 0 \pmod{p^4},$$

thus,  $b_2 \equiv 0 \pmod{p},$  (15)

$$b_2 = b_3 p.$$

It should be clear that we can iterate this procedure as many times as we like, without limit, to show that any power of  $p$  must divide  $b$ . Of course there is no finite number  $b$  satisfying such a constraint, and therefore our original hypothesis was in error. This proves that  $\cos(2\pi/N)$  and  $\sin(2\pi/N)$  cannot both be rational numbers, except for  $N = 2, N = 4$ , which were excluded initially. The proof holds for odd primes only, but the case of  $N$  a power of 2 is easily excluded separately.

This note has demonstrated mathematically what most people accepted intuitively, namely that the numerical evaluation of a  $z$  transform or a DFT by exact techniques is an impossibility.

## Loudspeaker System Figures of Merit

RICHARD H. SMALL

**Abstract**—The loudspeaker system small-signal figure of merit is a useful performance criterion that must be evaluated and used with some care. In practice, this figure of merit is significantly reduced by enclosure dissipation, including that contributed by filling materials.

The expression for small-signal figure of merit presented by Lampton [1] features an improvement on a similar expression developed by Small [2], [3], and some of the data presented lead to an important cautionary conclusion only lightly touched on by Lampton.

By separating the physical constant and the system-response factor  $K$ , Lampton provides a noteworthy improvement in comprehensibility and utility of the figure-of-merit expression compared to the form used by Small in which these are left combined as  $k_{\eta(G)}$  [3]–[5]. The improvement is twofold: first because  $K$  is a dimensionless number of comfortable magnitude that depends only on the system alignment, and second because  $K$  is independent of the system of units used (e.g., SI or English) and of the radiation angle of the system acoustic load; these conditions affect only the physical constant, which is easily adjusted for each situation.

There is a very interesting way of rearranging the basic efficiency expression that was pointed out to the author by Franklin Montgomery of the National Bureau of Standards following a lecture given at Washington, D.C., in June 1972. If the system half-power or  $-3$ -dB cutoff frequency  $f_3$  (Lampton's  $f_c$ ) is expressed in terms of its wavelength  $\lambda_3$ , the efficiency expression becomes

$$\eta_0 = 4\pi^2 K_{\eta} [V_B / \lambda_3^3] \quad (1)$$

where

- $\eta_0$  asymptotic efficiency for radiation into half-space (Lampton's  $E$ ),
- $K_{\eta}$  complete efficiency constant or figure of merit (Small's  $k_{\eta}$  divided by  $4\pi^2/c^3$ , or Lampton's product of  $K$ ,  $(1 - Q/Q_0)$ , and  $1.4/\gamma$ ), and
- $V_B$  net internal volume of the enclosure (Lampton's  $V$ ).

Every factor on the right-hand side of (1) is now dimensionless, including the final factor in brackets, provided that the enclosure dimensions and the wavelength are expressed in the same units.

In common with many other transducers and radiators, it is the small dimensions of the system in relation to a wavelength that restrict the radiation efficiency to a very low value; the magnitude of the bracketed factor for typical domestic high-fidelity sound reproducers is restricted to the order of  $10^{-4}$ .

The expressions for and values of the small-signal figure-of-merit response factors given by Lampton [1] and by Small [3]–[5] for both sealed enclosure and maximally-flat vented enclosure systems are in agreement when differences in normalization and notation are allowed for. It is particularly interesting, however, that the values of  $K$  for Lampton's bounded-ripple alignments exceed those reported by Small [3], [5] for the comparable Chebyshev alignments of the type used by Thiele [6]. The maximum value of  $K$  attainable with the Chebyshev alignments is about 4.1, occurring for a ripple of approximately 0.2 dB. (The Thiele alignment 9 used as an example by Lampton has about 0.5-dB ripple, not 1.8 dB as originally given by Thiele; errors in the ripple calculations were not discovered until after the publication of Thiele's paper.)

The difference in maximum  $K$  values arises from the fact that the Chebyshev alignments have positive ripple only, while the bounded-ripple alignments permit negative ripple as well. By extension, the "degenerated Chebyshev type I" alignments used by Nomura [7], which have negative ripple only, should exhibit even higher values of  $K$ . The "high- $K$ " alignment given by Lampton in [1] has this property, exhibiting 3 dB of negative ripple that depresses the response over a broad frequency range (Fig. 1). This alignment might well be subjectively disappointing, regardless of its high value of  $K$ . Lampton's goal of seeking alignments "for which the response curve is acceptable" thus deserves greater cautionary emphasis in relation to the additional goal of seeking a high figure of merit.

As pointed out by Lampton, alignments with ripple generally appear to offer a modest increase in figure of merit over flatter alignments. However, analysis of vented-box systems containing normal amounts of enclosure dissipation [3], [5] shows that this dissipation significantly reduces the figure of merit for all alignments but especially for alignments with ripple. The result is effectively a loss of the figure-of-merit advantage of these alignments.

It should be noted that the statements in [1] concerning the effects of enclosure filling material are somewhat oversimplified and could lead to difficulty or disappointment in practical applications. The factor  $\gamma$  in the denominator of [1, eq. (4)] is seldom reduced by more than 20 percent when a practical filling material is added to a loudspeaker enclosure. In addition,  $\gamma$  is not the only factor that may be affected. Depending on alterations made to the system at the time of filling (such as reducing  $V$  to keep  $S$  constant as  $\gamma$  falls), the material and its losses also generally affect  $S$ ,  $G$ , the effective value of  $Q_0$  and thus  $Q$ .

The overall effects of filling material on the small-signal figure of merit for sealed systems are exhibited more clearly when the efficiency expression is renormalized in terms of system parameters. This procedure is followed in [4], where the effects are analyzed and discussed in some detail; a very useful large-signal figure of merit is also developed in this paper.

Filling materials are seldom employed in vented enclosures. The effects are analyzed in [3, ch. 7], where it is shown that

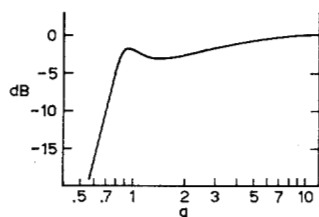


Fig. 1. Normalized frequency response of the Lampton "high-K bass reflex" alignment.

the advantage of increased enclosure compliance is usually more than offset by the losses introduced by the material. This analysis is included in a paper to be published in the near future.

The statements in Lampton's conclusion regarding loudspeaker (driver) size represent only one possible interpretation of his (5). Apparently, a smaller value of  $S$  (for fixed  $V$ ) has been interpreted as implying a smaller value of  $A_1$ , perhaps on the assumption that  $s_1$  is fixed. Actually, it is the entire factor  $A_1^2/s_1$  that is constrained to be smaller. Thus  $A_1$  may have any desired value within the limits of physical realizability of  $s_1$  [2], [8], [9].

In this respect, the bounded-ripple and Chebyshev alignments

are quite similar. Compared to the "triple-point" or Butterworth alignment, both require a stiffer diaphragm suspension for a given constraint on enclosure volume and diaphragm area. Both are therefore less vulnerable to very low frequency driving signals and can exhibit better large-signal output characteristics [10].

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